Splitting planar graphs of bounded girth to subgraphs with short paths

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An (a,b)-colouring is acyclic if it has no monochromatic cycles, i.e. every monochromatic component is a tree of diameter at most $a–1$ or $b–1$ respectively.

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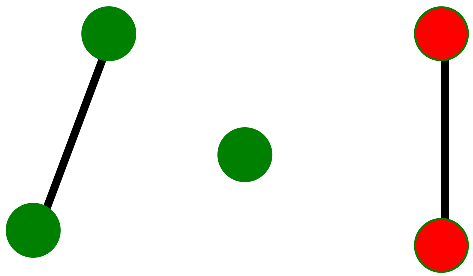
(2,2)-colouring ≡ monochromatic components are K_1 and K_2



Possible components for acyclic (7,7)-colouring

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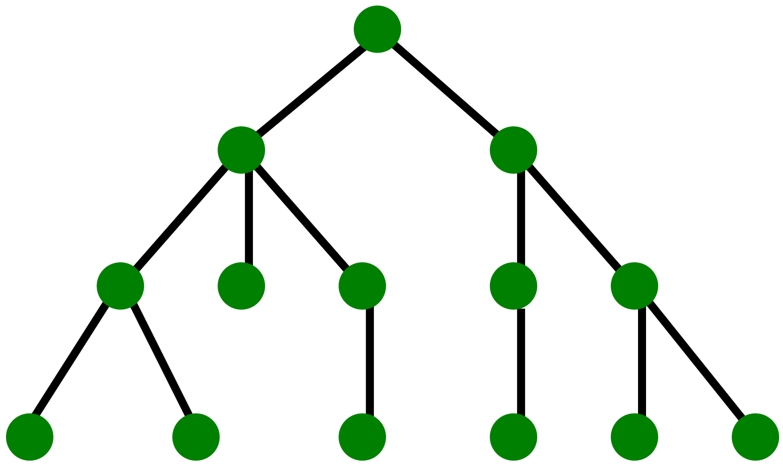

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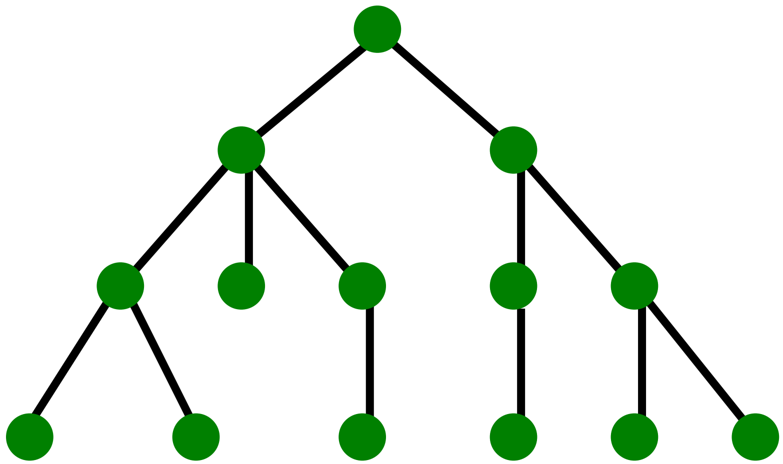

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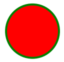

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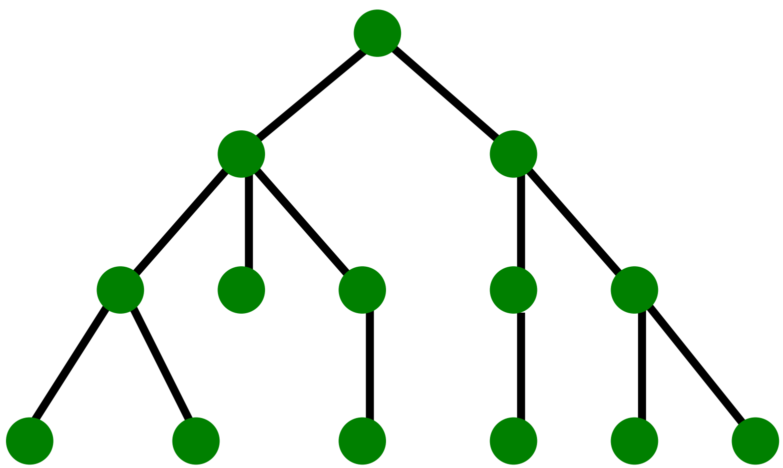

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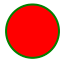
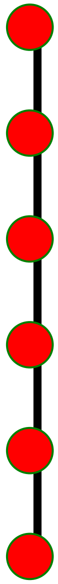


Path



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Question: What is the minimum integer $g_0 > 4$ such that every planar graph of girth at least g_0 is (a,b)-colourable for some constants a,b?

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M. Axenovich, T. Ueckerdt, P. Weiner (2015):

Every planar graph of girth ≥ 6 has an acyclic (15,15)-colouring such that every monochromatic component is a path (colouring by linear forests).

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Theorem 1a (list version): Every planar graph of girth ≥ 5 is list acyclically (7,7)-colourable.

(Every vertex *v* gets its colour from a list *L*(*v*) of size |*L*(*v*)| = 2.)

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Thomassen's List 5-colour Theorem:

Every planar graph is 5-choosable.

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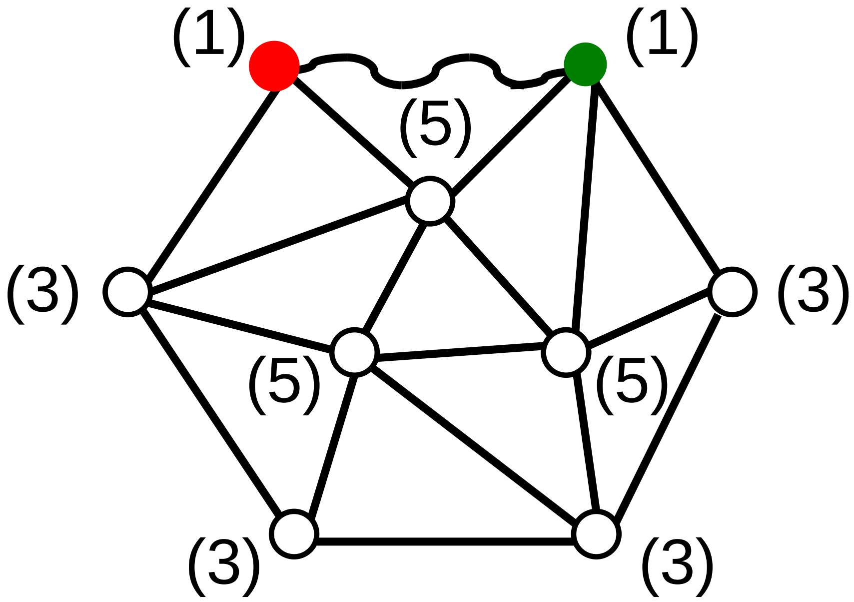
Technical Theorem: Every planar graph is *L*-colourable for any list assignment *L* satisfying:



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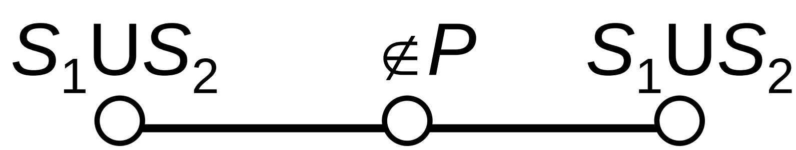
4) G has no path:



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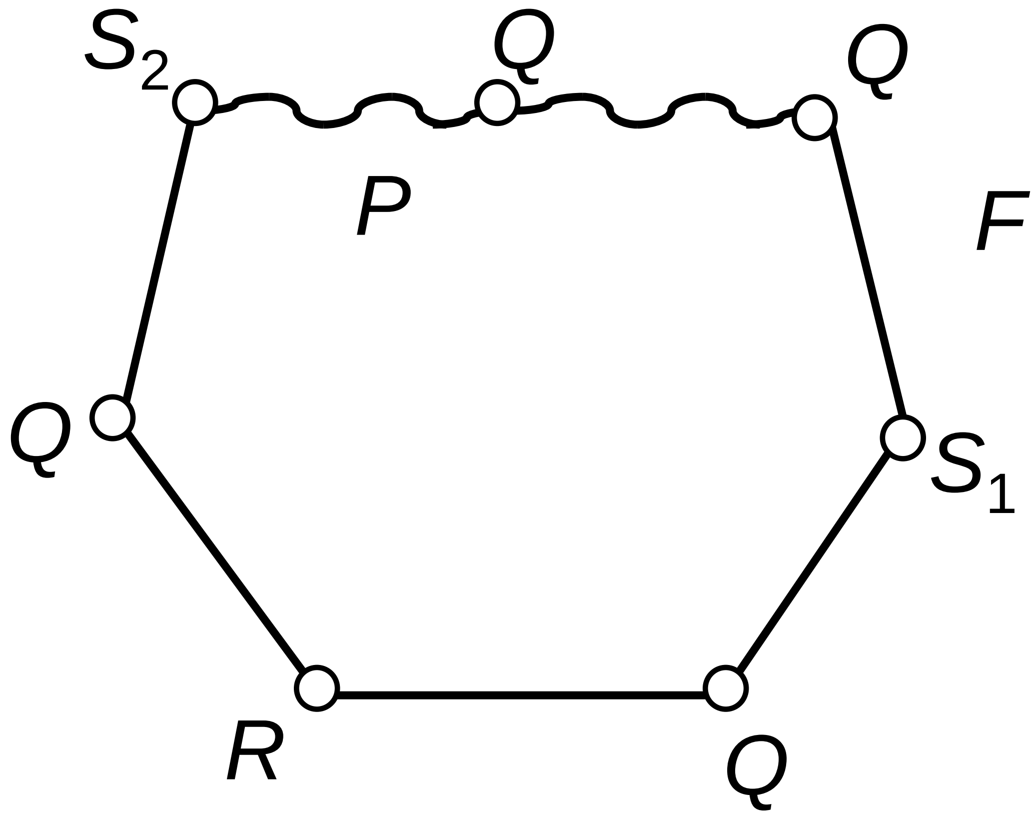
Marking $M = (P,Q,R,S_1,S_2)$ of F (*example*):



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2) $\forall v \in R$ if T is a mon. component containing v, then the rooted tree T_v has height $h(T_v) \leq 2$:

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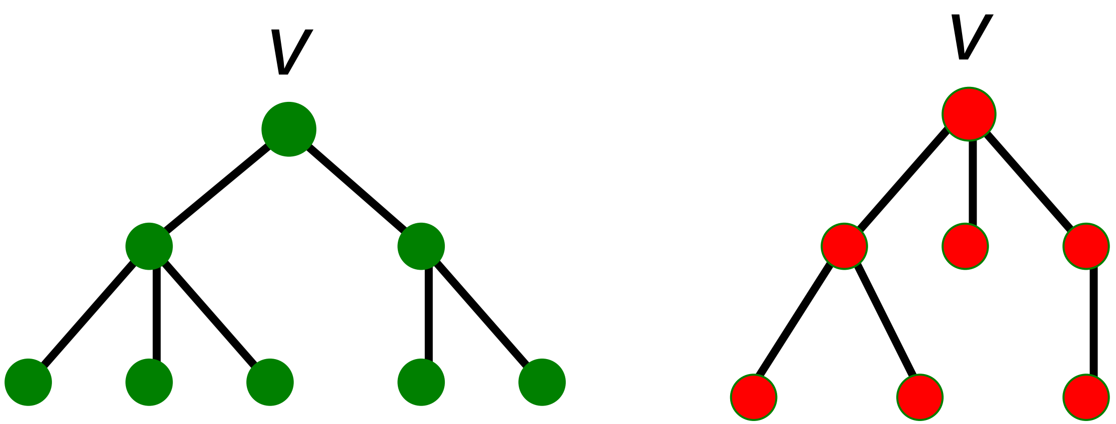

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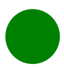

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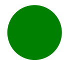

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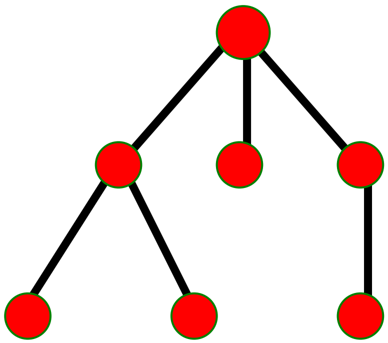

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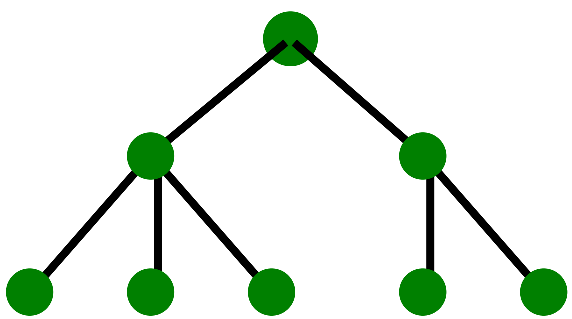

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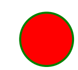

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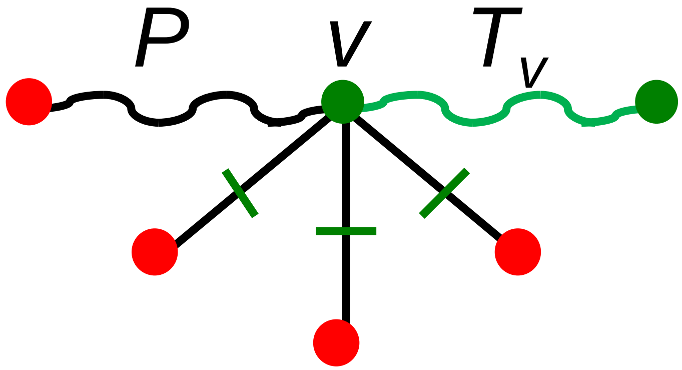
b) if $v \in P$, then $T_v \subseteq P$, i.e.



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Theorem 2: Suppose $G = (V,E)$ is a connected plane graph of girth $g(G) \geq 5$; F is an outer face of G; $M = (P,Q,R,S_1,S_2)$ is a marking of F. Then any (pre)colouring $\varphi_P : V(P) \to \{1,2\}$ of P which fits M can be extended to an acyclic $(7,7)$-colouring of G fitting M.

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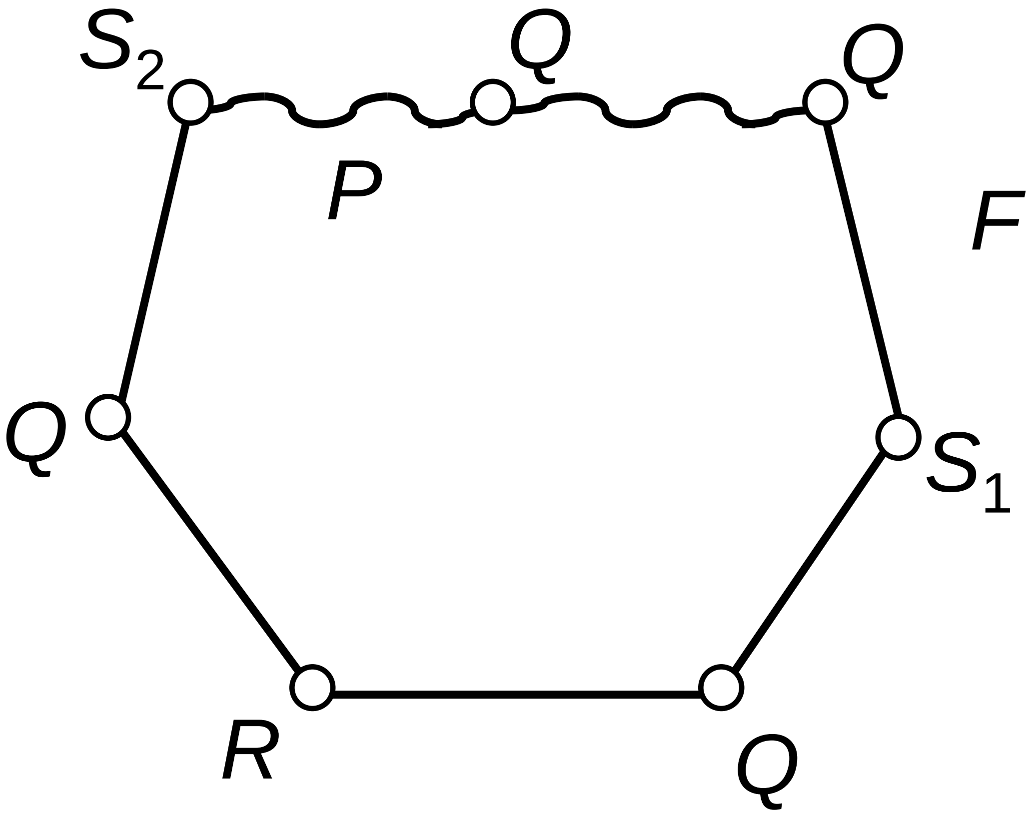
Theorem 2 $\Rightarrow$ Theorem 1

Main technical result



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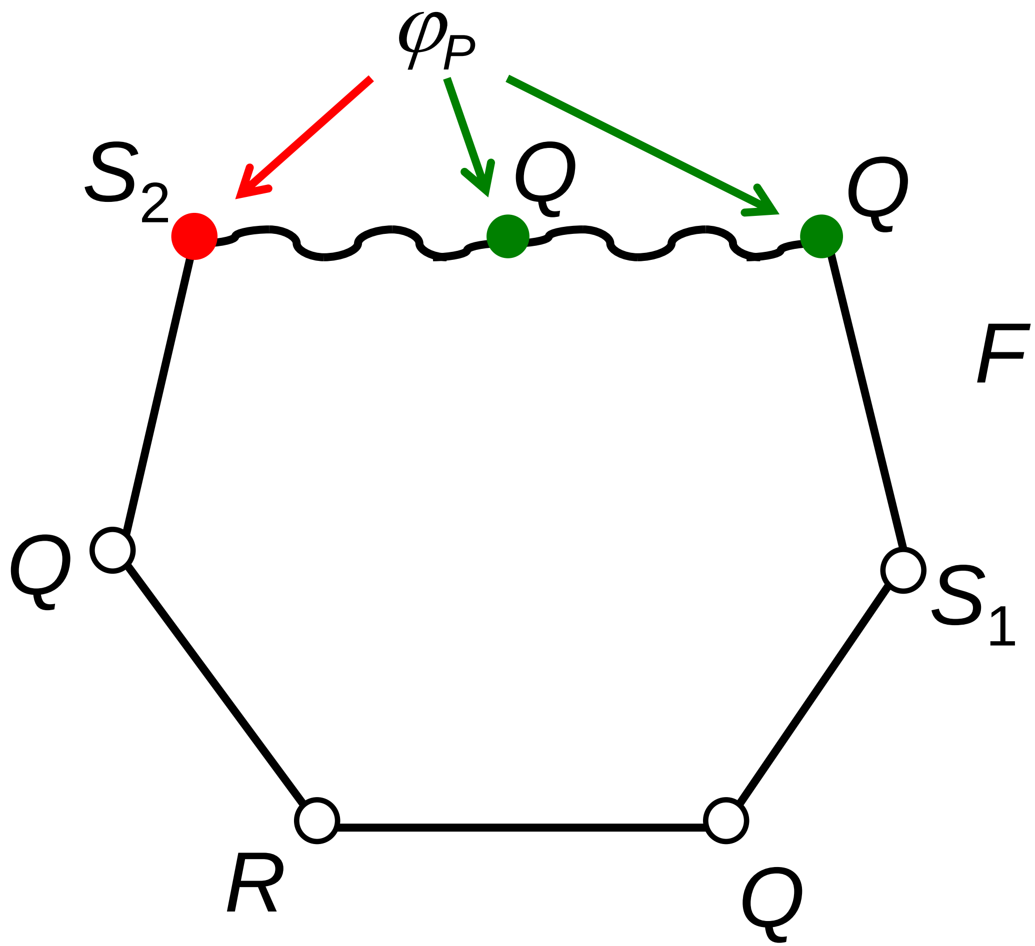

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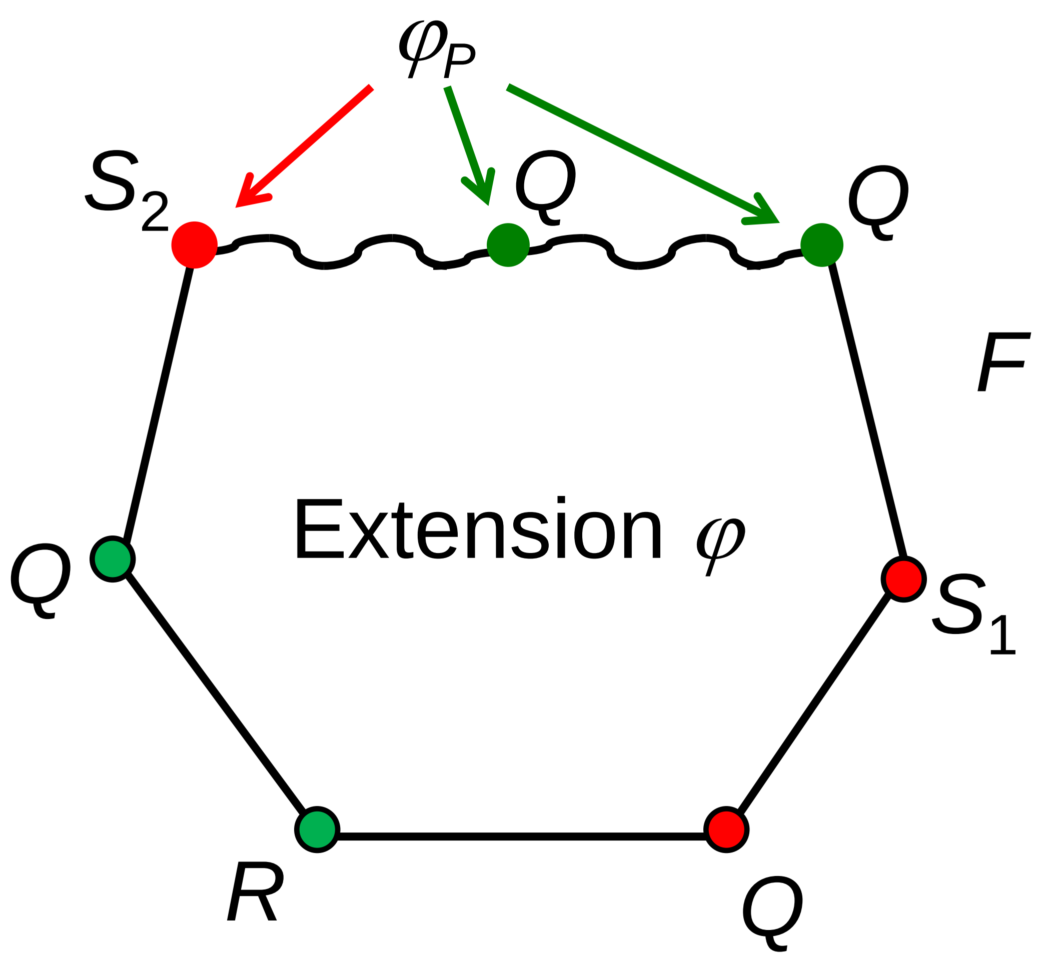
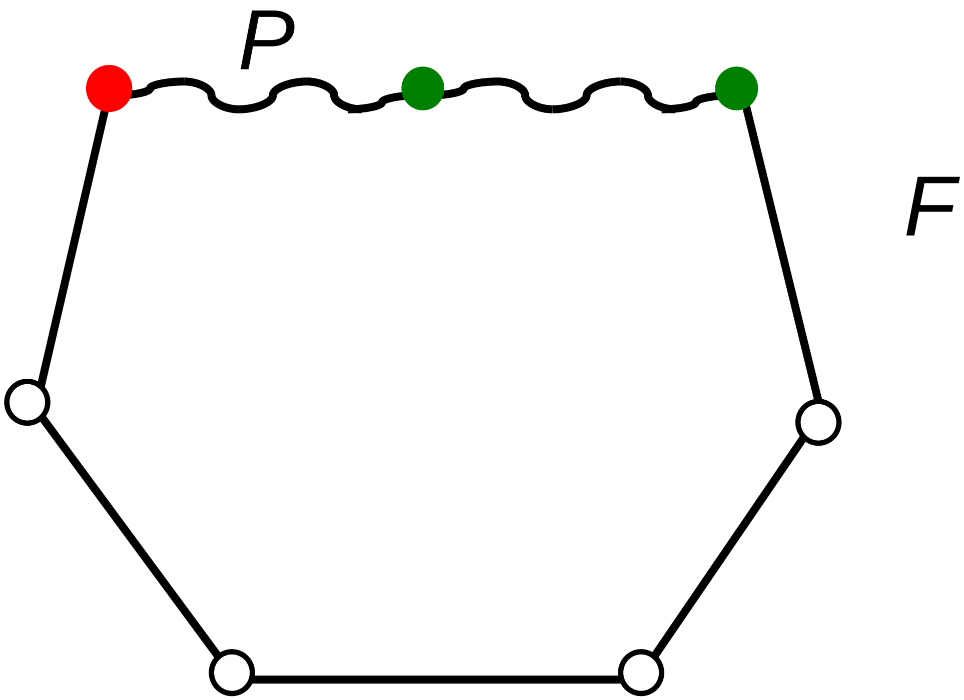


Thank you for your kind attention!

Precoloured path P

A path P of length at most 2 on the boundary of F whose vertices are precoloured by colours 1 and 2 (green and red): $\varphi_P : V(P) \to \{1, 2\}$.



2 Peripatetic Salesman Problem with distances 1 and 2.

Input:

Complete undirected graph: $G = (V, E)$

Weight function: $w : E \to \{1,2\}$

Solution:

Two edge-disjoint Hamiltonian cycles:

$$H_1,\ H_2 \quad : \quad H_1 \cap H_2 = \varnothing$$

$$w(H_1 \cup H_2) \ \to \ \min$$

6/5-approximation algorithm $A_{6/5}$ for 2-PSP(1,2)

(E. Gimadi, Y. Glazkov, A. Glebov, 2007)

Stage 1. By Gabow's Algorithm find a 4-regular subgraph

$$G_4 \leq G: \quad w(G_4) \to min$$

Stage 2. By Algorithm $P_{4/5}$ find two edge-disjoint partial tours T_1 and T_2 in G_4: $|E(T_1 \cup T_2)| \geq 4/5\,|E(G_4)| = 8n/5$

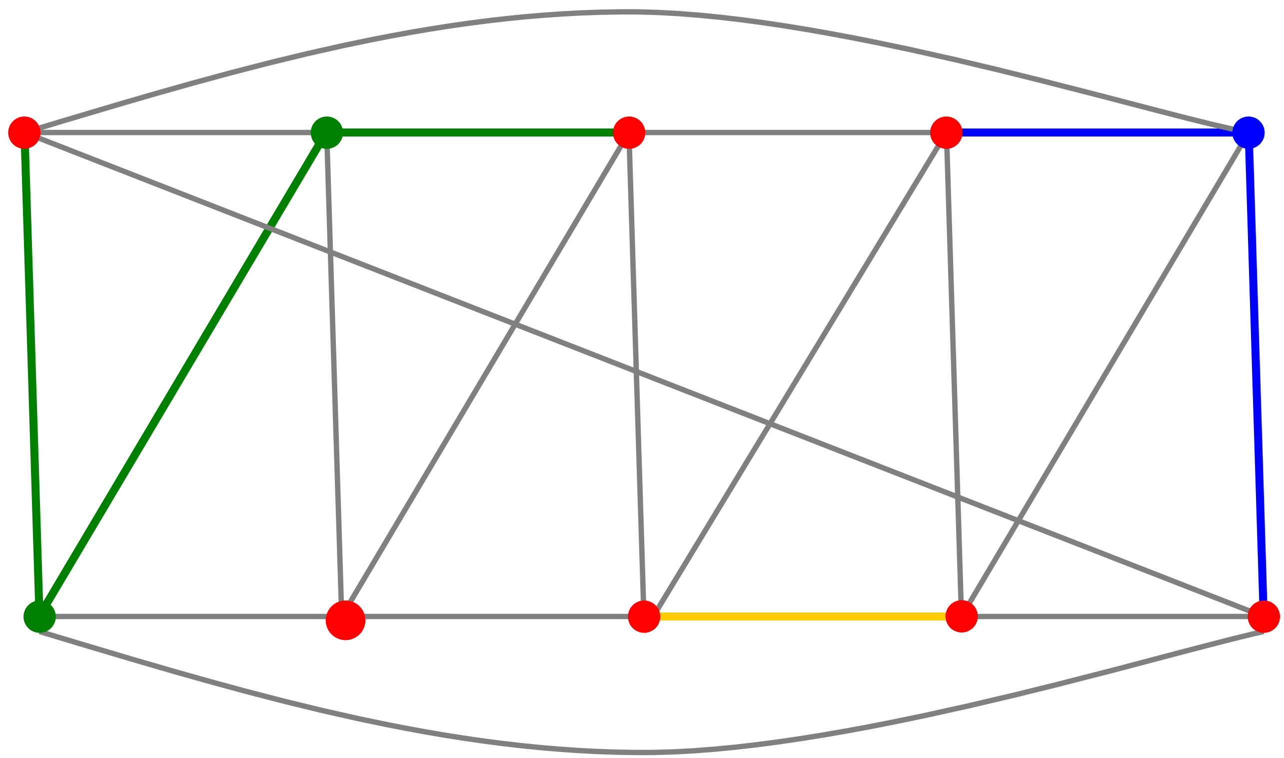
Stage 3. Extend T_1 to a Hamiltonian cycle H_1.

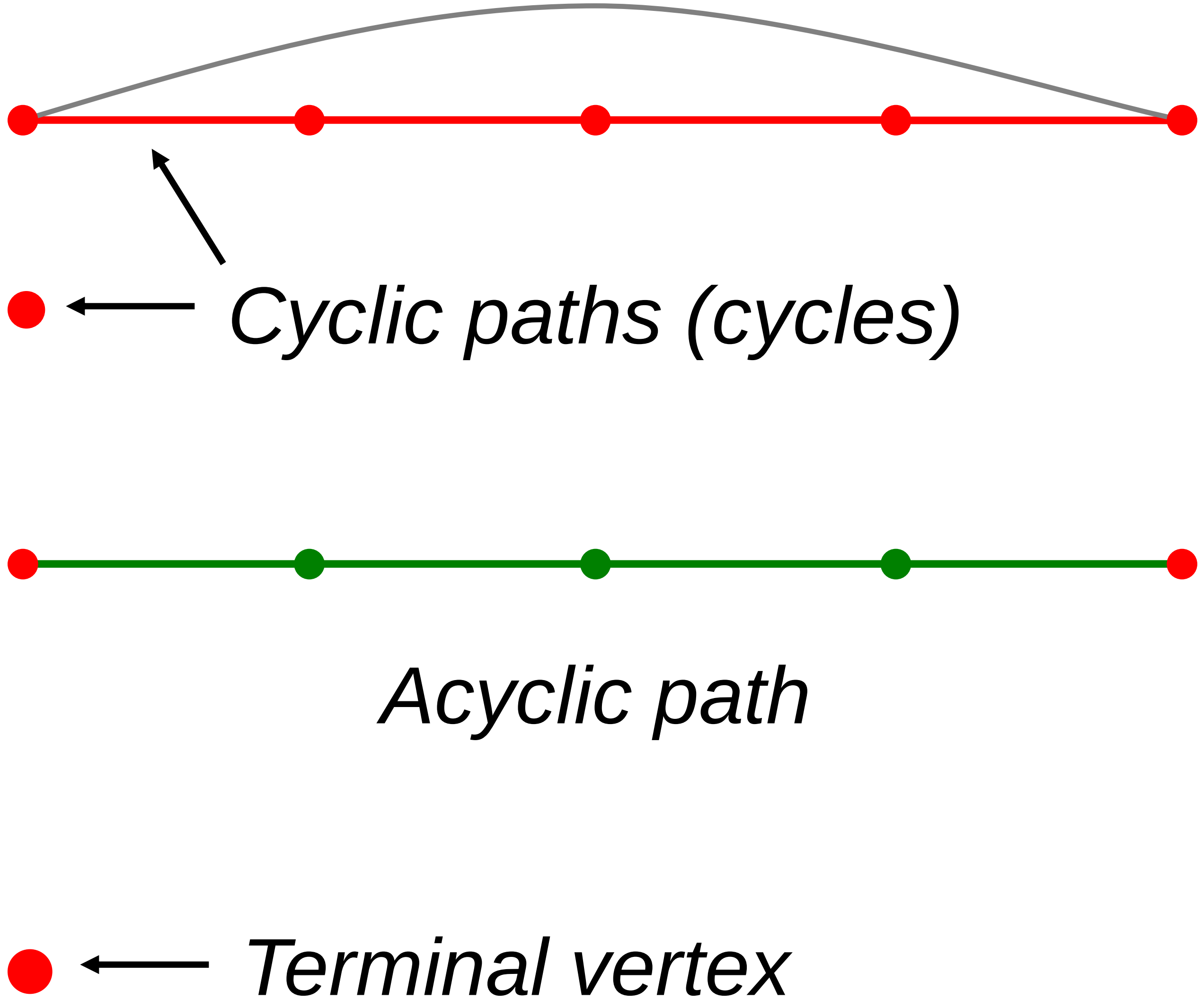
By Procedure $P_{T\to H}$ extend T_2 to a Hamiltonian cycle H_2 :

-

$$H_1 \cap H_2 = \varnothing$$

(Partial) Tour – a collection of vertex-disjoint paths of a graph containing all its vertices





Cyclic paths (cycles)
Acyclic path
Terminal vertex

Algorithm $P_{4/5}$ for finding two partial tours in a 4-regular graph G_4

Stage 1. Constructing T_1

Step 1.1. Find initial T_1 (by any algorithm):

K – number of paths;

C – number of cyclic paths:

If $C = 0 \rightarrow$ Stage 2;

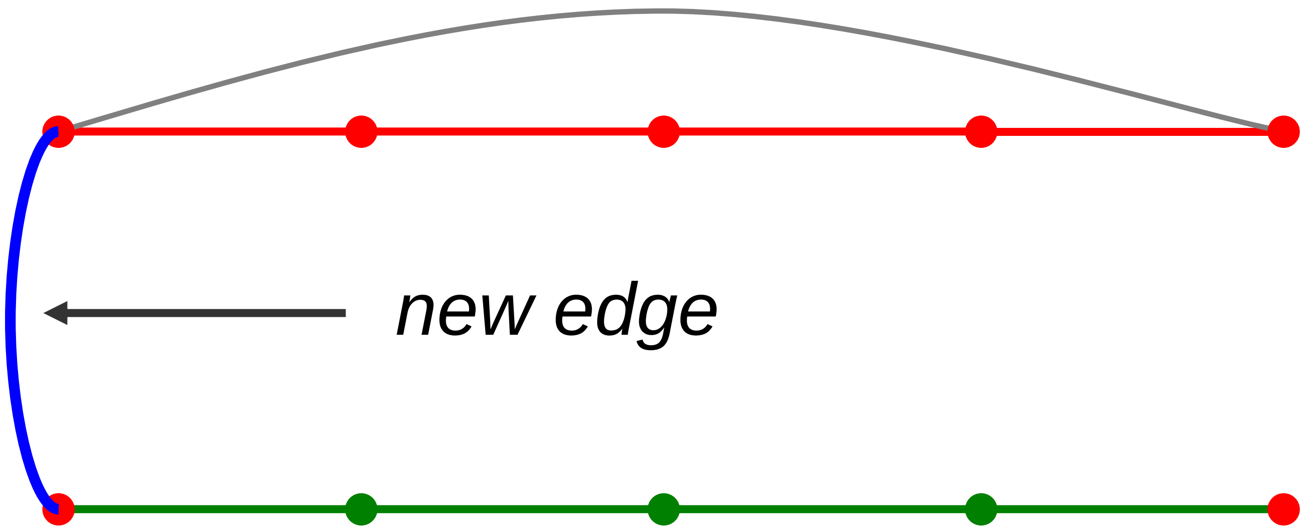
$Q = 2K + C$ – *quality* of T_1

Step 1.2. Joining terminal vertices

If $C = 0 \rightarrow$ Stage 2;

If terminal vertices are nonadjacent $\rightarrow$ Step 1.3;

Otherwise:

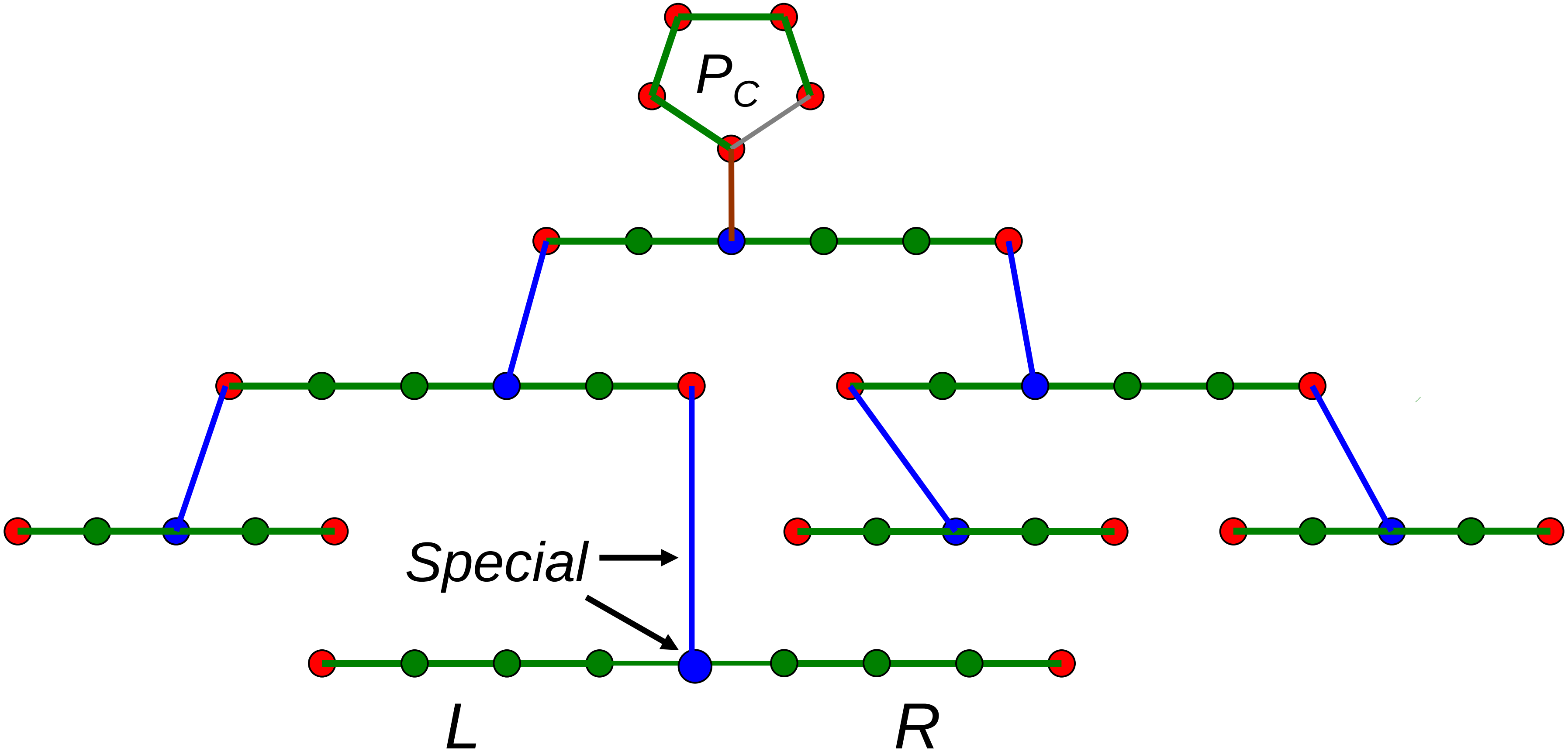


$$Q = 2K + C \rightarrow (Q - 2) \text{ or } (Q - 1)$$

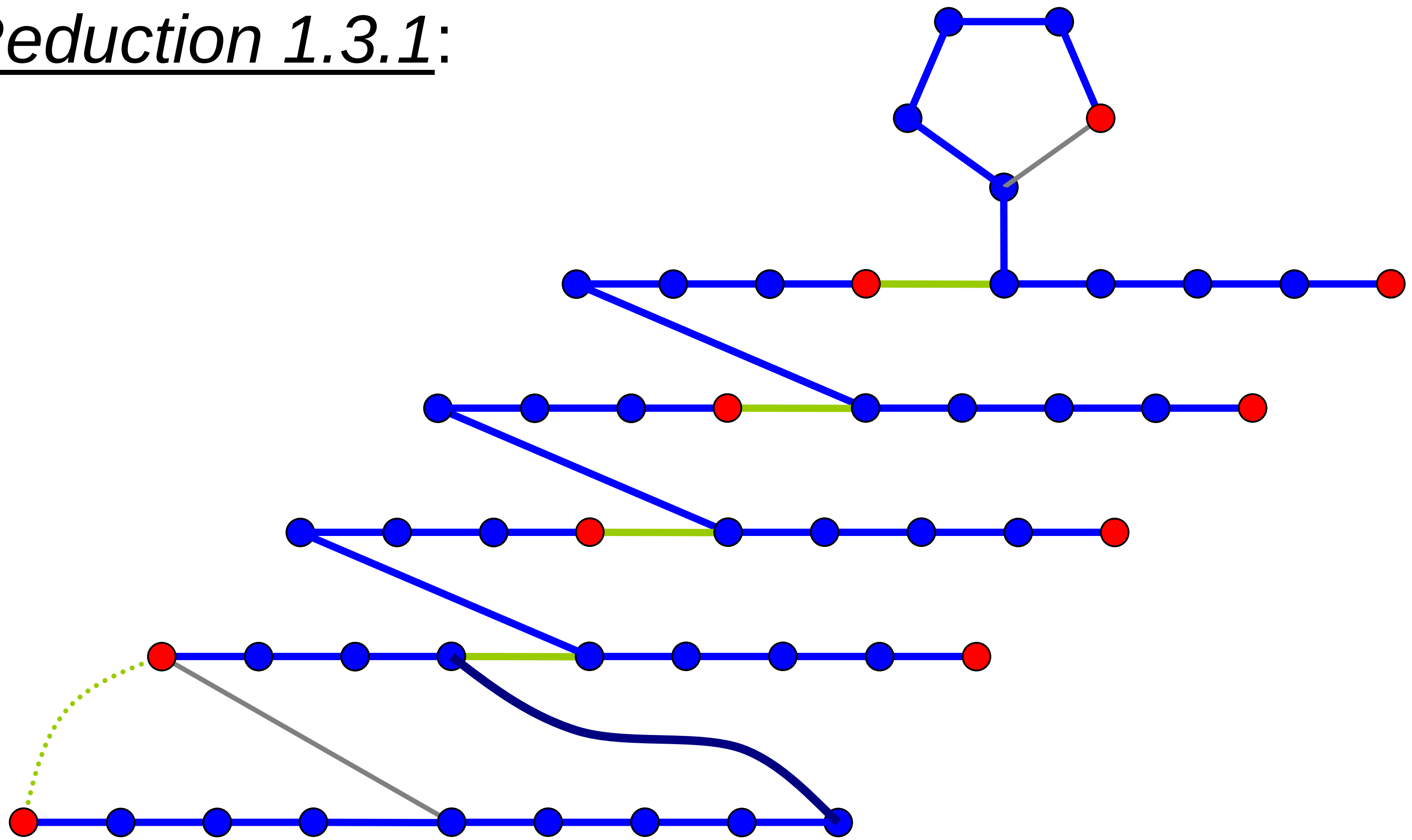
Step 1.3. Subtracting cyclic paths

If $C = 0 \rightarrow$ Stage 2; Otherwise

1) find cyclic path P_C; 2) construct a *Path Tree F*:

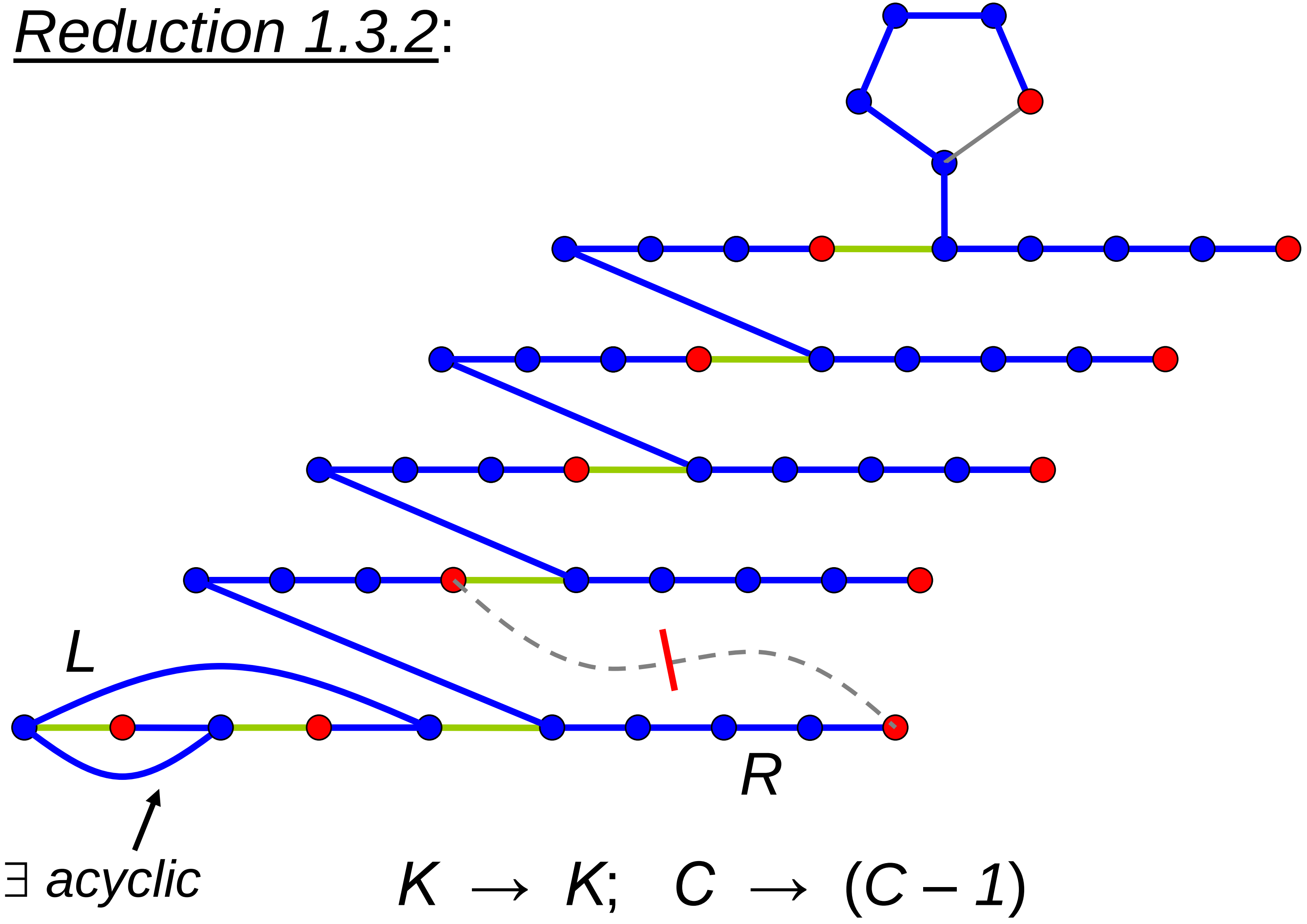


Reduction 1.3.1:



$K \rightarrow (K-1);\quad C \rightarrow C$

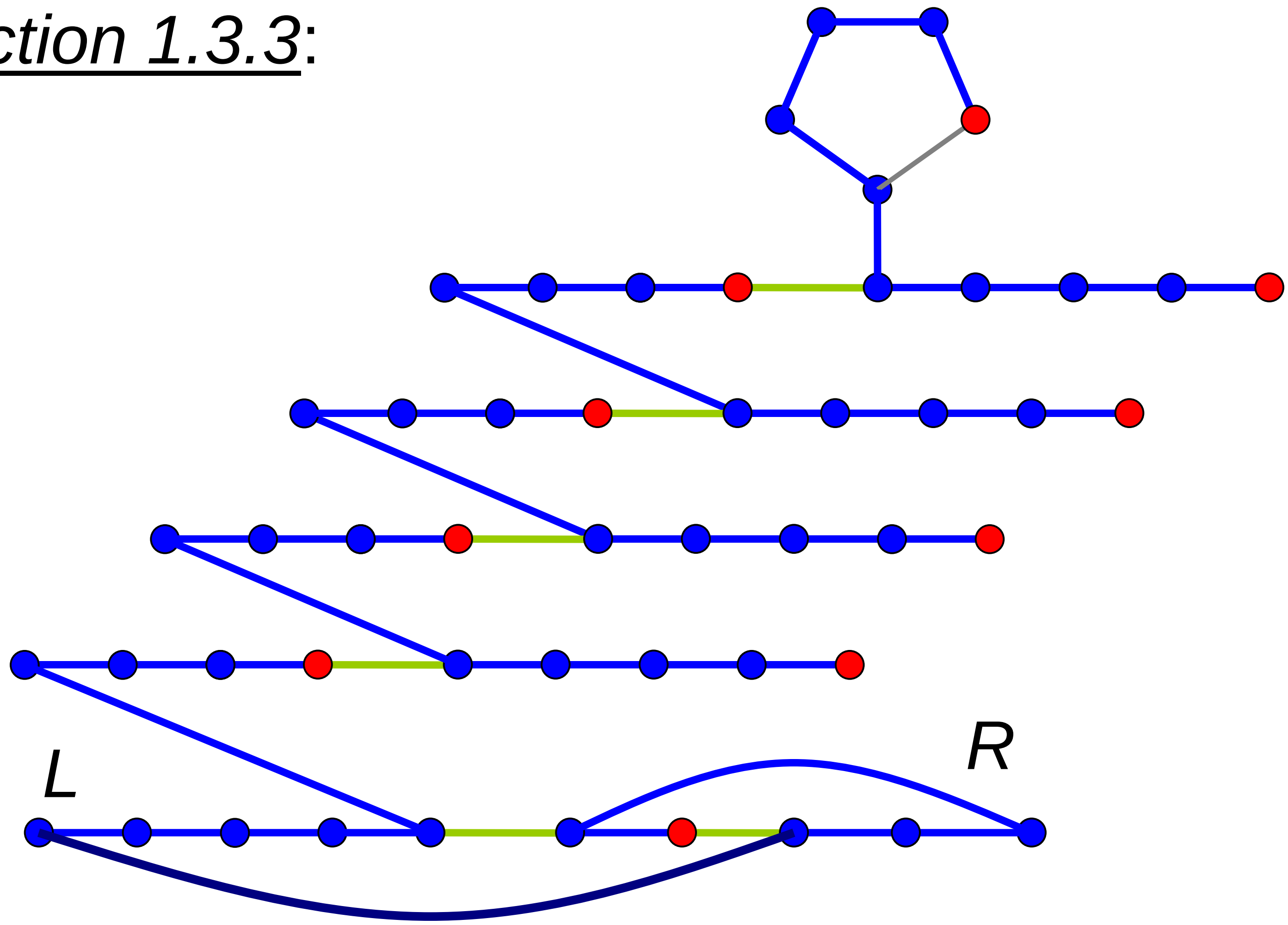
Reduction 1.3.2:



$K \rightarrow K; \quad C \rightarrow (C - 1)$

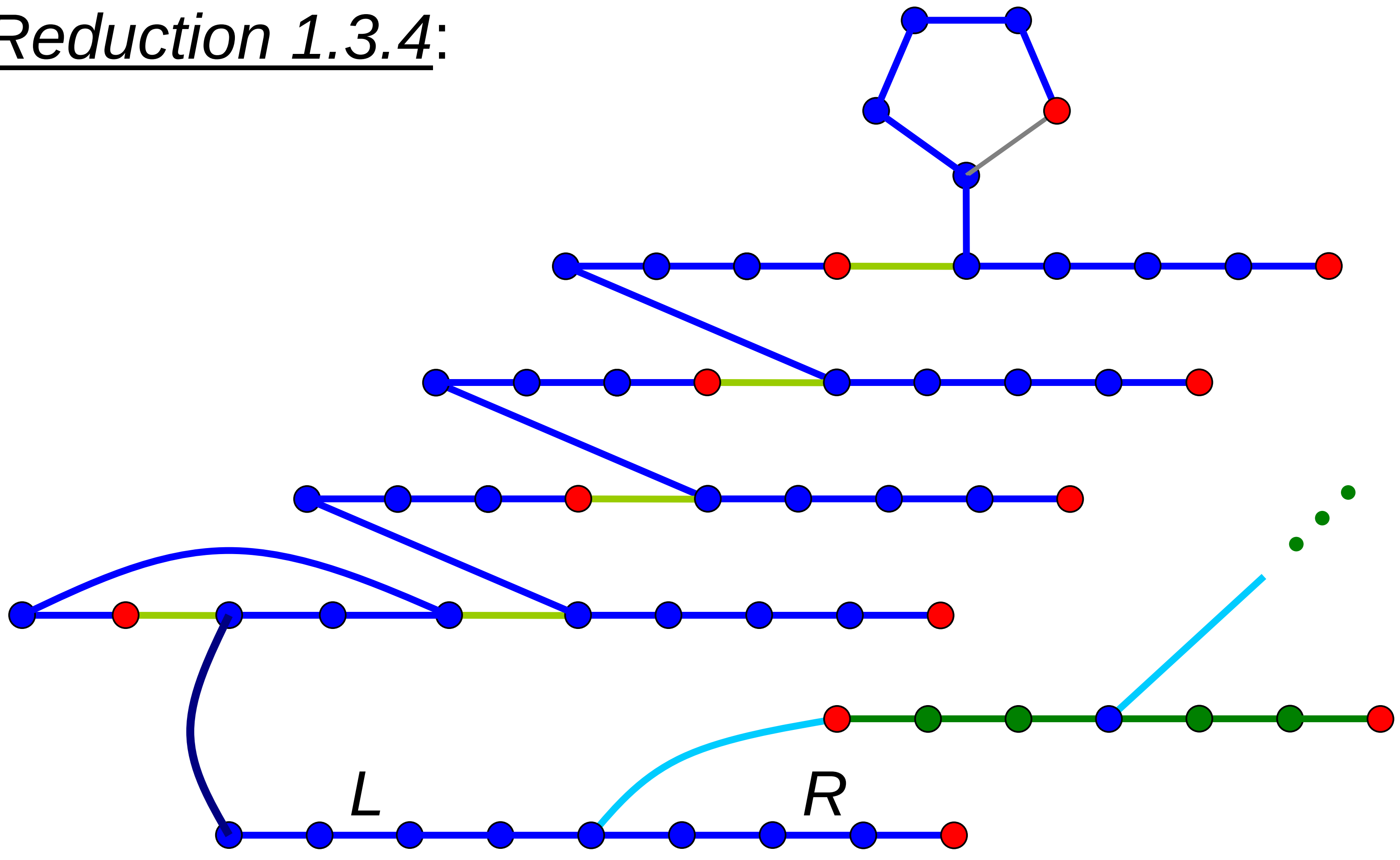
<u>Procedure P_{AC}.</u> If L (or R) is not isomorphic to C_m, K_1, K_4 or $K_{3,3}$, then we find an acyclic Hamiltonian path in it.

Reduction 1.3.3:



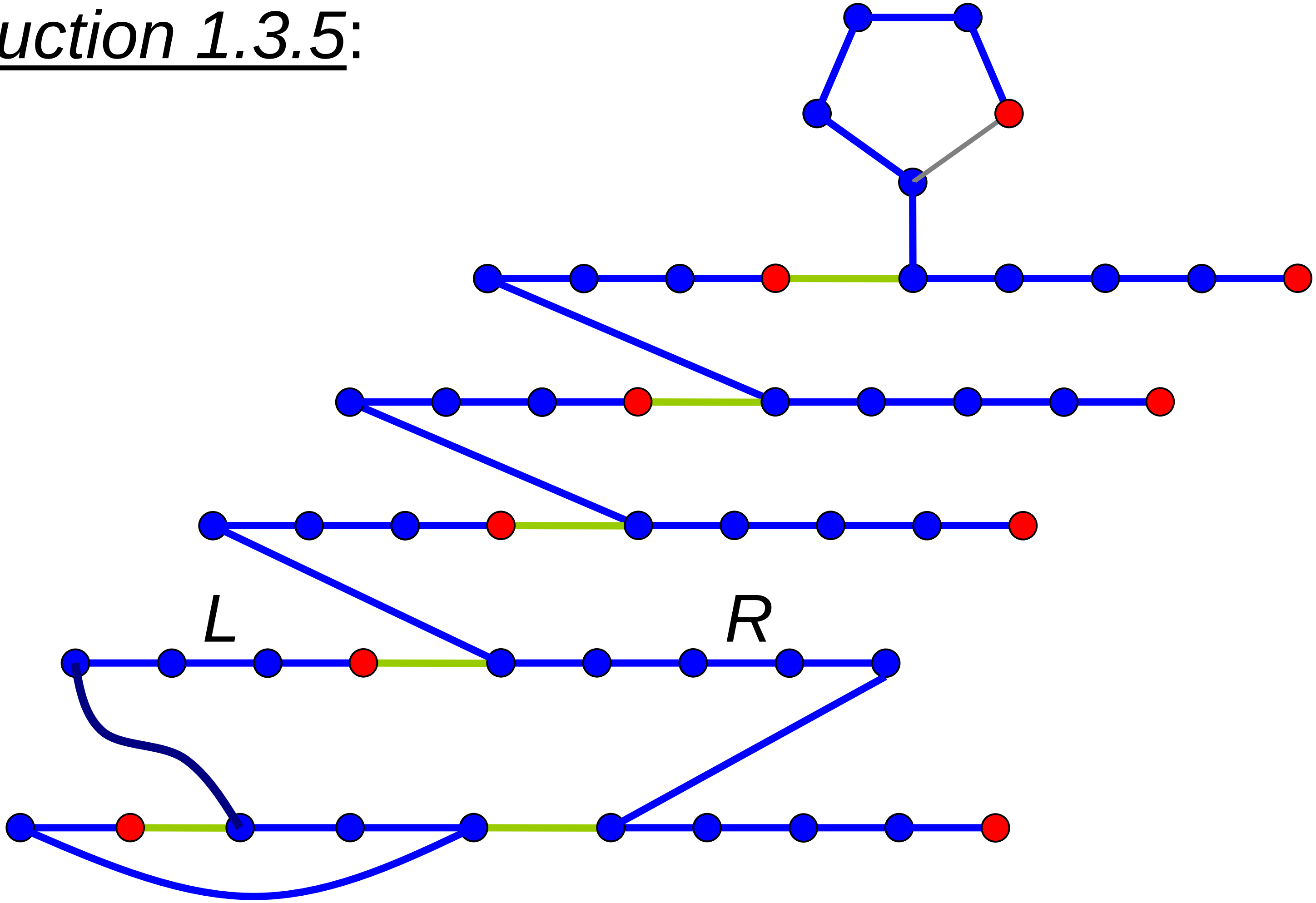
$$K \rightarrow (K-1); \quad C \rightarrow C$$

Reduction 1.3.4:



$$K \rightarrow (K-1); \quad C \rightarrow C$$

<u>*Reduction 1.3.5*</u>:



$$K \rightarrow (K - 1); \quad C \rightarrow C$$

After making a *Reduction* $\rightarrow$ Step 1.2

Lemma. If G_4 is not isomorphic to K_5 or $K_{4,4}$, then after Stage 1 we get a partial tour T_1 with $K \le n/5$ and $C = 0$, in which terminal vertices are nonadjacent.

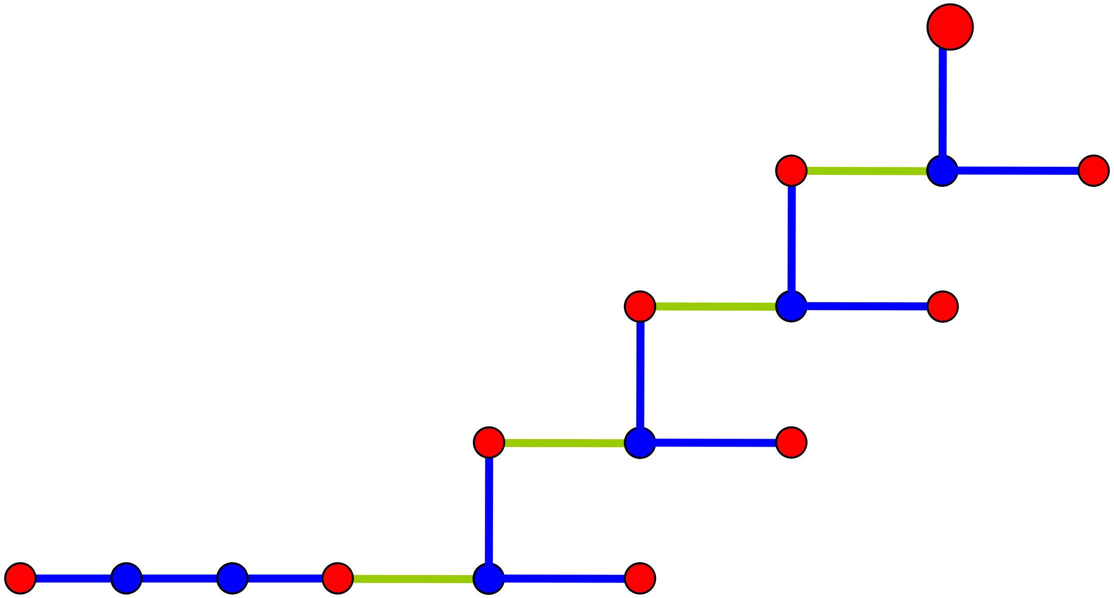
Stage 2. Constructing T_2

Step 2.0. $H := G - T_1$ – a subgraph in G_4 with $2K$ vertices of degree 3 and $n - 2K$ vertices of degree 2, in which 3-vertices are nonadjacent.

Step 2.1. Find initial T_2 in H (any algorithm)

Step 2.2. Join terminal vertices of T_2 (analog of Step 1.2)

Step 2.3. Subtracting single-vertex paths (simplified Step 1.3):



After every *reduction* $\rightarrow$ Step 2.2

Lemma. After Stage 2 the partial tour T_2 has $K = K(T_1)$ acyclic paths and at most $(n - 5K)/3$ cyclic paths.

Theorem 1. $P_{4/5}$ is a quadratic time algorithm which finds edge-disjoint partial tours T_1 and T_2 in G_4: $|E(T_1 \cup T_2)| \geq 8n/5$.

$T_1 \rightarrow H_1$;

$T_2 \rightarrow H_2$ by Procedure $P_{T \rightarrow H}$:

$$H_1 \cap H_2 = \varnothing$$

<u>*Theorem 2*</u>. $A_{6/5}$ is a cubic time 6/5-approximation algorithm for 2PSP(1,2):

$$w(H_1 \cup H_2) \leq 6/5\, w(OPT).$$